SIDH and its Applications

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Evolution of Di e-Hellman

$$\xrightarrow{\quad g^a \quad}$$

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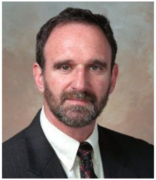

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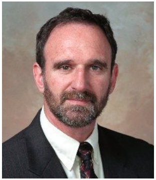

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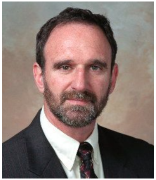

$$\xrightarrow{[a]P}$$

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$$\xrightarrow{{}_A(E_0)}$$

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credits: Craig Costello's ECC2017 talk

Isogenies

Definition

Let E, E' be two elliptic curves, and let $\varphi : E \to E'$ be a map between them. φ is called an *isogeny*, if

- φ is a surjective group homomorphism
- φ is a group homomorphism with finite kernel
- φ is a non-constant rational map with $\varphi(O_E) = O_{E'}$

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- For any finite subgroup $H \subset E$, there exists an isogeny $\varphi : E \to E' := E/H$ with kernel H
- For (separable) isogenies, $\#\ker(\varphi)$ is the degree of φ

Factoring isogenies

De nition (Universal property)

Let $\varphi : E \to E'$ be an isogeny. If $P \in \ker(\varphi)$, then there exist isogenies $\;\;;\;\;$ such that $\ker(\;\;) = \langle P \rangle$ and

$$\varphi = \qquad$$

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De nition (j-invariant)

Let $E : y^2 = x^3 + ax + b$. Then, the j-invariant of E is

$$j(E) := 1728 \quad \frac{4a^3}{4a^3 + 27b^2} \; 2 \; \mathrm{F}_{p^2}:$$

Supersingular isogeny graphs

Definition (ℓ-isogeny graph)

The supersingular ℓ-isogeny graph over $\mathbb{F}_{p^2}$ consists of

- vertices are j-invariants of supersingular elliptic curves defined over $\mathbb{F}_{p^2}$
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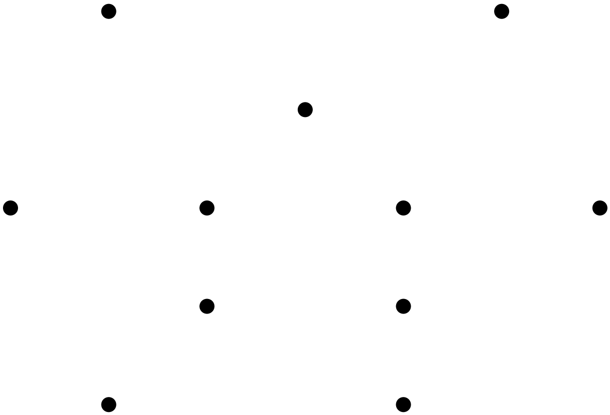
- path finding is postulated to be exponentially hard both classically and quantumly

SIDH [JD11]

Idea: Alice and Bob walk in two *different* isogeny graphs on the *same* vertex set.



2- and 3-isogeny graph on $\mathbb{F}_{127^2}$

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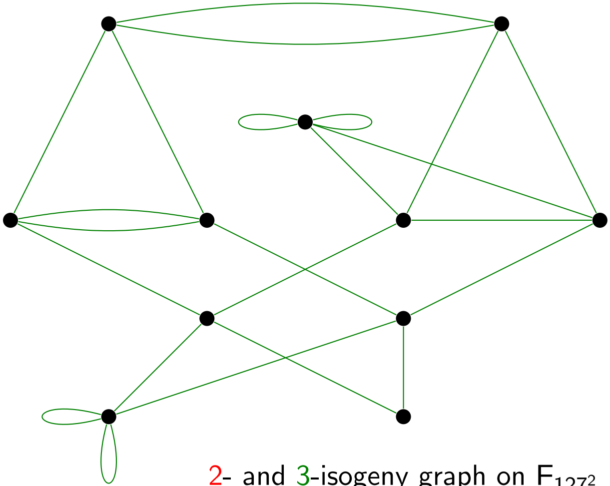

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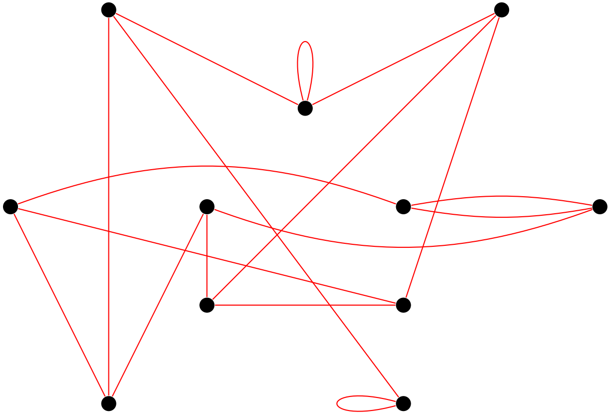

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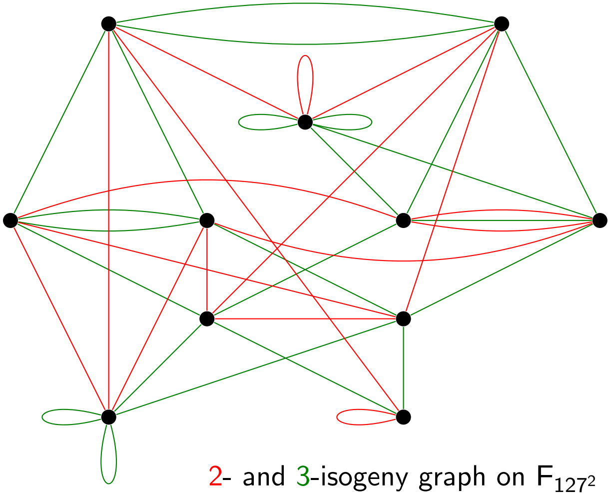

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SIDH (cont.)

- Fix a prime p such that $p = N_1 N_2 - 1$, E_0=$\mathbb{F}_p^2$ and bases $\langle P_A, Q_A \rangle = E_0[N_1]$, $\langle P_B, Q_B \rangle = E_0[N_2]$

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  $A := P_A + [\mathrm{sk}_A] Q_A$
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- The shared secret is the j-invariant of E_{AB}

- In SIDH the secret isogenies are relatively short $(N_1 \approx N_2 \approx \sqrt{p})$
- Between two randomly chosen supersingular elliptic curves an isogeny of this degree does not exist in general

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- Main idea of B-SIDH: Use isogenies such that $N_1 \quad N_2 \quad p$ and $p^2 \quad 1 = N_1 N_2 f$
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- Keys are even smaller than in SIDH but it is also slower as $N_1; N_2$ are less smooth

Hard problems

Definition (Pure isogeny problem)

Given two isogenous supersingular elliptic curves E_1 and E_2, compute an isogeny $\phi : E_1 \to E_2$.

Definition (SSI-T Problem)

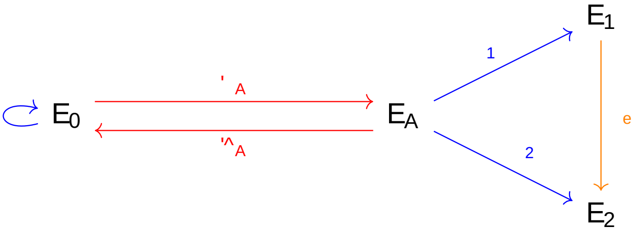
Let $\phi : E_1 \to E_2$ of degree N_1 and let $E_1[N_2] =: \langle P; Q \rangle$. Given E_1, E_2, $\phi(P)$, $\phi(Q)$, compute ϕ.

Torsion point attacks [Pet17, QKL+21]

Target: SSI-T, when End(E_0) is known.

$\varphi_A : E_0 \to E_A$ implies

$$Z + \varphi_A \qquad \hat{\varphi}_A \to \mathrm{End}(E_A)$$



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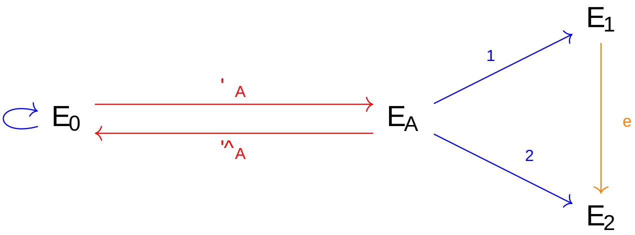

- Consider $= [d] + \varphi_A \qquad \hat{\varphi}_A$ and assume $\deg() = N_2^2 e$
- Compute $= \hat{}_2 \quad e \quad {}_1$ using torsion point information and small meet-in-the-middle search

Torsion point attacks (cont.)

Once $= [\,d] + \varphi_A \qquad \hat{\varphi}_A$ is known:

$$\ker \hat{\varphi}_A = \ker(\qquad [d]) \cap E_A[N_A]$$

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To break SSI-T problem, su cient to nd d and such that

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- For $j(E_0) = 1728$, this yields norm equation

$$d^2 + N_1^2 \;\; c^2 + p \;\; b^2 + a^2 \quad = N_2^2 e$$

- Know how to nd solutions when $N_2 > pN_1$.

Seta [DDF+21]

Previous work [Pet17, QKL+21, KMPW21]:

- $E_0 : y^2 = x^3 + x$ or E_0 can be chosen by the adversary
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Seta public key encryption:

- backdoor curve used as public parameter
- message corresponds to an isogeny from this starting curve
- ciphertext contains codomain of isogeny and torsion point images
- decryption performed using torsion point attacks

SIKE: Supersingular Isogeny Key Encapsulation

- Submission to NIST's PQ standardisation process:
  - SIKE.PKE: El Gamal-type system with IND-CPA security proof
  - SIKE.KEM: generically transformed system with IND-CCA security
- Smallestcommunication complexity for each of the security levels (1,3,5)
- Slowestamong all proposals for each of the security levels
- https://sike.org/

Static-static key exchange

Can we use SIDH as a drop-in replacement for Di e-Hellman?

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Post-quantum version of Signal's initial X3DH key exchange:

- SI-X3DH [DG21]3

Goal: Prove knowledge of secret isogeny φ of degree $\ell_1^{e_1}$.

De Feo-Jao-Plût scheme: Let $E_0[\ell_2^{e_2}] = \langle P_0, Q_0 \rangle$. (E_0, P_0, Q_0) are public parameters and $(E_1, \varphi(P_0), \varphi(Q_0))$ are the public key.

1. Prover generates randomly $\ell_2^{e_2}$-torsion point $K_\psi := [a]P_0 + [b]Q_0$ corresponding to $\psi : E_0 \to E_2$

$$\begin{tikzcd} E_0 \arrow[r, "\varphi"] \arrow[d, "\psi"] & E_1 \arrow[d, "\psi'"] \\ E_2 \arrow[r, "\varphi'"] & E_3 \end{tikzcd}$$

and commits to E_2 and E_3.

Proof of Isogeny Knowledge

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2. Verifier challenges the prover with a random bit $c \in \{0, 1\}$
3. Prover reveals (a, b), if $c = 0$, and $\psi(\ker(\varphi))$, if $c = 1$.

Proof of Isogeny Knowledge (cont.)

Soundness issue of De Feo-Jao-Plût scheme [DDGZ21]:

$$
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- Prover generates random isogeny $\psi' : E_3 \to E_1$ of degree $\ell_2^{e_2}$ and picks P_0', Q_0' such that $\ker(\hat{\psi}') = [a]P_0' + [b]Q_0'$
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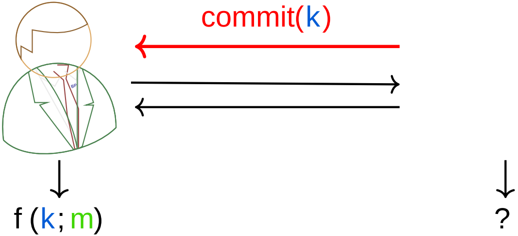
- Prover can respond to all challenges
- Isogeny $E_0 \to E_1$ of degree $\ell_1^{e_1}$ will not exist in general

Oblivious Pseudorandom Function (OPRF)

An OPRF is a two-party protocol to evaluate a PRF $f(k; m)$ where:

- The client learns $f(k; m)$, one evaluation of a PRF on a chosen input
- The server learns nothing about m



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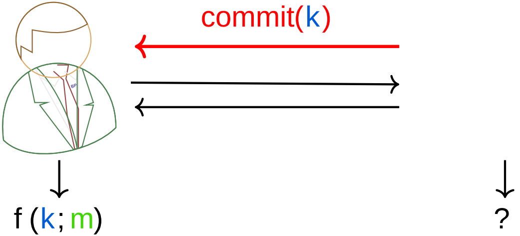

- An OPRF is called *verifiable*, if the server proves to the client that output was computed using the key k

Existing Constructions

Parameters: group G of order q, hash functions H_1, H_2 onto G and $\{0,1\}^\ell$ resp.

Client $C(m)$		Server $S(k)$
Pick $r \leftarrow_R \mathbb{Z}_q$		
Set $a \leftarrow (H_1(m))^r$	$\xrightarrow{a}$	
		If $a \in G$, set $b \leftarrow a^k$
	b	
If $b \in G$, set $v \leftarrow b^{1/r}$		
Output $H_2(m, v)$		

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Post-quantum OPRF:

- Construction from lattices [ADDS19]
- Construction from isogenies [BKW20]

Oblivious Pseudorandom Functions from Isogenies [BKW20]

Client

Server

$$\begin{tikzcd}
E_0 \arrow[r, "m"] \arrow[d] \arrow[dr] & E_M \arrow[d] \arrow[dr, "r"] & \\
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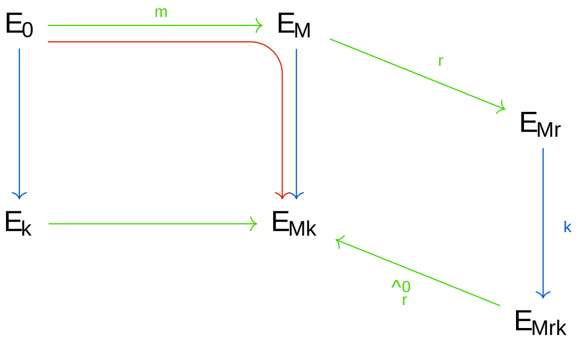

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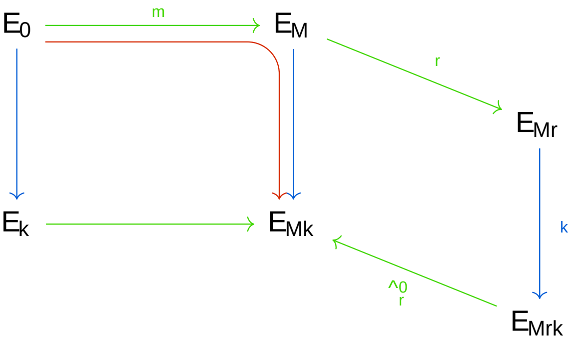
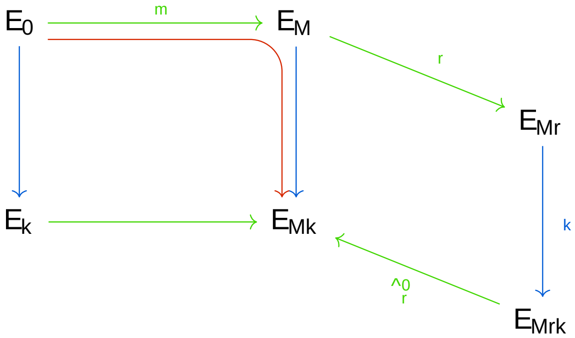
Server



Oblivious Pseudorandom Functions from Isogenies [BKW20]

Client

Server



Oblivious Pseudorandom Functions from Isogenies [BKW20]

Client

Server

$$\begin{tikzcd}
E_0 \arrow[r, "m"] & E_M \arrow[dr, "r"] & \\
& & E_{Mr} \arrow[dd, "k"] \\
E_k & E_{Mk} & \\
& & E_{Mrk} \arrow[ul, "\hat{r}'"]
\end{tikzcd}$$

Oblivious Pseudorandom Functions from Isogenies [BKW20]

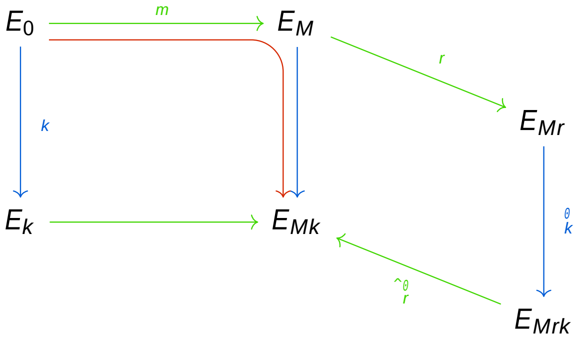
$$
\begin{tikzcd}
E_0 \arrow[r, "m"] \arrow[ddr, bend left] & E_M \arrow[dr, "r"] & \\
 & & E_{Mr} \arrow[dd, "k"] \\
E_k & E_{Mk} & \\
 & & E_{Mrk} \arrow[ul, "\hat{r}^{0}"]
\end{tikzcd}
$$

Client

Server

$$f(k; m) = H(m; j(E_{Mk}); \mathrm{pk})$$

Attacking the Pseudorandomness [BKM+21]



- Use queries to the OPRF to obtain E_k and $_k(E_0[2^n])$ up to scalar multiplication

Attacking the Pseudorandomness [BKM+21]

$$\begin{tikzcd}
E_0 \arrow[r, "\phi_m"] \arrow[dd, "\phi_k"] \arrow[rdd, bend left] & E_M \arrow[dd] \arrow[rd, "\phi_r"] & \\
 & & E_{Mr} \arrow[dd, "\phi'_k"] \\
E_k \arrow[r] & E_{Mk} & \\
 & & E_{Mrk} \arrow[luu, "\hat{\phi}'_r"]
\end{tikzcd}$$

- Use queries to the OPRF to obtain E_k and $\phi_k(E_0[2^n])$ up to scalar multiplication
- Given $P \in E_0[2^n]$, compute $\langle \phi_k(P) \rangle$ and $E_k / \langle \phi_k(P) \rangle = E_{Pk}$

Updateable Public-Key Encryption

Desired properties:

- Correctness
- Forward secrecy
- Post-compromise security
- Asynchronicity
- Key indistinguishability

Updateable Public-Key Encryption from SIDH [EJKM20]

$$
\begin{tikzcd}
E_0 \arrow[r, "\varphi_{A_1}"'] \arrow[d, "\psi_B"] & E_{A_1} \arrow[d, "\psi'_B"'] \\
E_B \arrow[r, "\varphi'_{A_1}"] & E_{A_1 B}
\end{tikzcd}
$$

Updateable Public-Key Encryption from SIDH [EJKM20]

$$
\begin{tikzcd}
E_0 \arrow[r, "\varphi_{A_1}"'] \arrow[d, "\varphi_B"] & E_{A_1} \arrow[r, "\varphi_{A_2}"'] \arrow[d, "\varphi'_B"'] & E_{A_2} \arrow[d, "\varphi''_B"'] \\
E_B \arrow[r, "\varphi'_{A_1}"] & E_{A_1B} \arrow[r, "\varphi'_{A_2}"] & E_{A_2B}
\end{tikzcd}
$$

Updateable Public-Key Encryption from SIDH [EJKM20]

$$
\begin{tikzcd}
E_0 \arrow[r, "'_{A_1}"'] \arrow[rr, bend left, "'"] \arrow[d, "_B"] & E_{A_1} \arrow[r, "'_{A_2}"'] \arrow[d, "'_B"'] & E_{A_2} \arrow[d, "''_B"'] \\
E_B \arrow[r, "''_{A_1}"] & E_{A_1 B} \arrow[r, "''_{A_2}"] & E_{A_2 B}
\end{tikzcd}
$$

Idea: Use [KLPT14] to compute $'$ from $'_{A_2}$ $'_{A_1}$ to achieve post-compromise security and forward secrecy

Updateable Public-Key Encryption from SIDH [EJKM20]

$$
\begin{tikzcd}
E_0 \arrow[r, "'_{A_1}"'] \arrow[rr, bend left, "'"] \arrow[d, "_B"] & E_{A_1} \arrow[r, "'_{A_2}"'] \arrow[d, "^{0}_{B}"'] & E_{A_2} \arrow[d, "^{00}_{B}"'] \\
E_B \arrow[r, "'^{0}_{A_1}"] & E_{A_1B} \arrow[r, "'^{0}_{A_2}"] & E_{A_2B}
\end{tikzcd}
$$

Idea: Use [KLPT14] to compute $'$ from $'_{A_2}$ $'_{A_1}$ to achieve post-compromise security and forward secrecy

Caveats:

- Very unbalanced parameters
- No asynchronicity
- No key indistinguishability

Conclusion

- SIDH has small keys and is reasonably fast
- Some advanced cryptographic protocols from SIDH exist
- Many subtle issues when building schemes from SIDH
- Further work is required:
  - Find new isogeny-based protocols
  - Remove limitations of existing constructions
    (e.g. sample supersingular elliptic curves without revealing their endomorphism ring)
  - Cryptanalyse existing constructions

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