One-way functions and malleability oracles: Hidden shift attacks on isogeny-based protocols

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Overview

- We find a *quantum* subexponential attack on (overstretched and unbalanced) SIDH via reduction to an injective abelian hidden shift problem.
- For ordinary curves and CSIDH: Childs-Jao-Soukharev give a quantum subexponential attack to compute an isogeny between two curves using this reduction.
- Previously widespread belief for SIDH: *"Since the algorithm of Childs et al. depends crucially on the properties of abelian groups, we believe that no reasonable variant of this strategy would apply to supersingular curves."* [DJP11]
- **Disclaimer:** The attack does not apply to balanced SIDH or SIKE parameters.

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Isogenies

Let E_0, E_1 be elliptic curves defined over a field K.

- An *isogeny* is a non-constant rational map $\varphi : E_0 \to E_1$ that is also a group homomorphism.
- (Separable) isogenies correspond to finite subgroups of E_0.
- The kernel of an isogeny determines the image curve up to isomorphism. $(E_0 / \ker(\varphi) := E_1)$
- Two curves E_0, E_1 are isomorphic if and only if they have the same j-invariant.

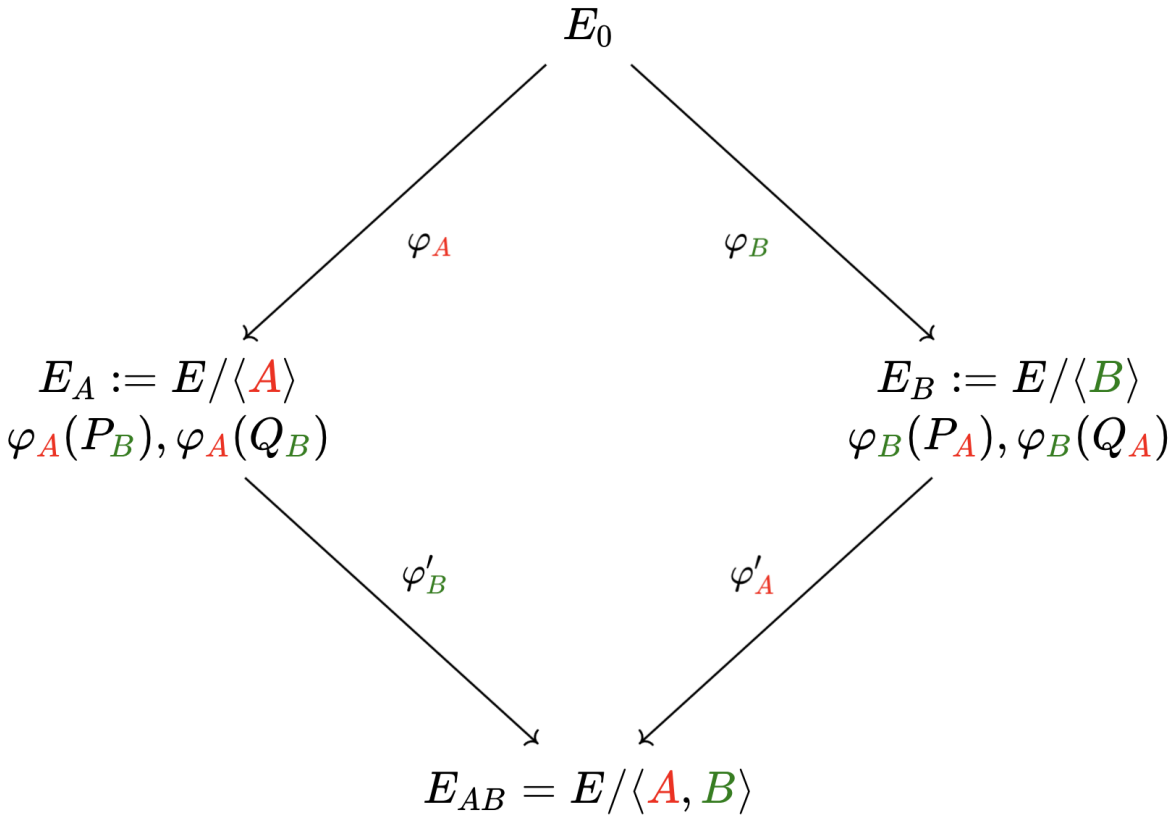
Isogeny-based cryptography

- Candidate for post-quantum cryptography.
- Based on the hardness of finding (large degree) isogenies between supersingular elliptic curves.
- Most isogeny-based cryptosystems (e.g. SIDH) are based on a relaxation of this problem.
- Small key sizes.

SIDH [JD11]

- Fix a prime p such that $p = N_1 N_2 - 1$, $E_0/\mathbb{F}_p^2$ and bases $\langle P_A, Q_A \rangle = E_0[N_1]$, $\langle P_B, Q_B \rangle = E_0[N_2]$.

- Alice's secret is $A := P_A + [\mathsf{sk}_A] Q_A$.
- Bob's secret is $B := P_B + [\mathsf{sk}_B] Q_B$.



- Alice sends E_A, $\phi_A(P_B)$, $\phi_A(Q_B)$.
- Bob sends E_B, $\phi_B(P_A)$, $\phi_B(Q_A)$.

- The shared secret is the j-invariant of E_{AB}.

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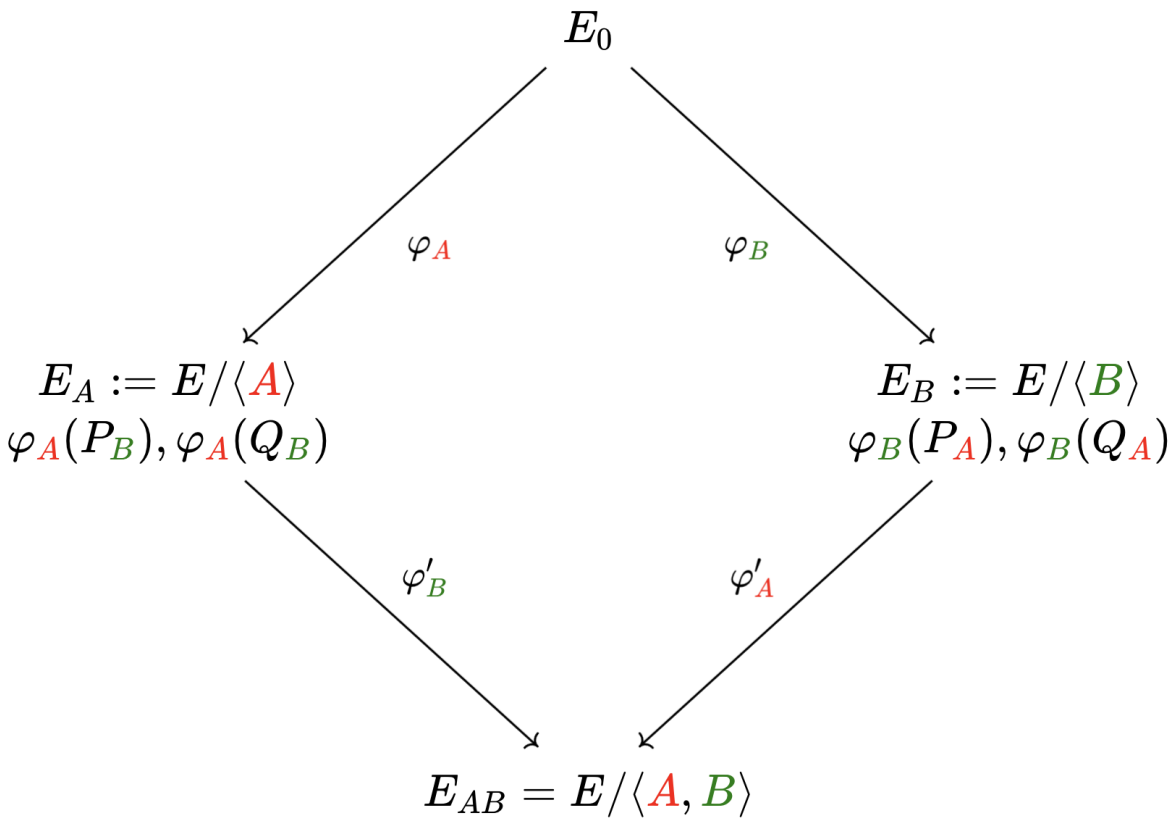

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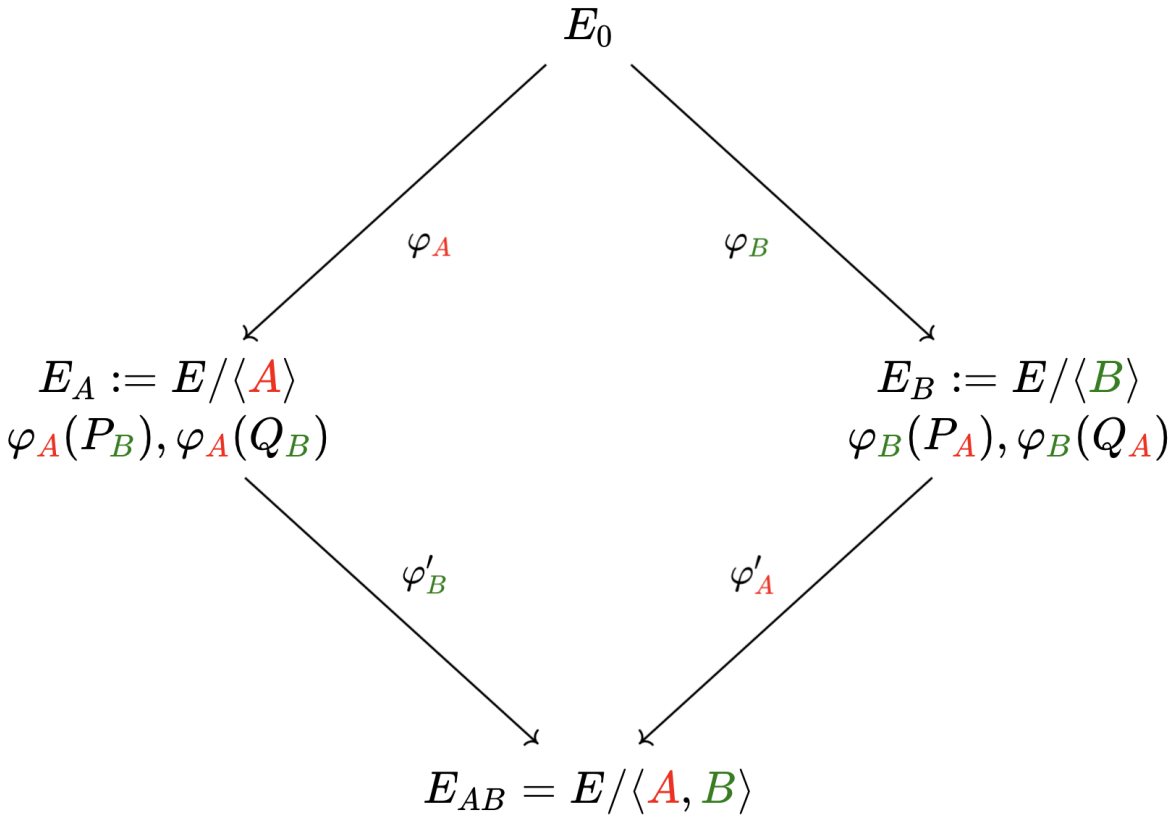

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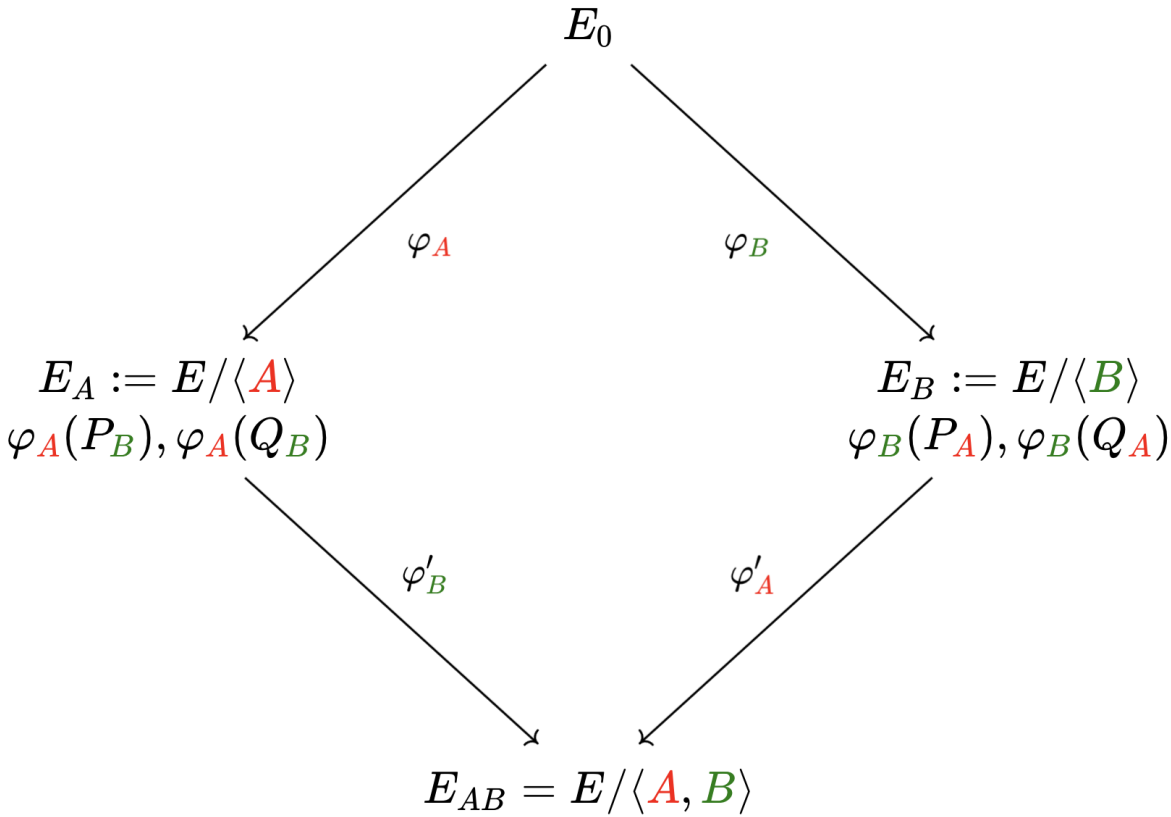

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The hidden shift problem

Definition (Hidden shift problem)

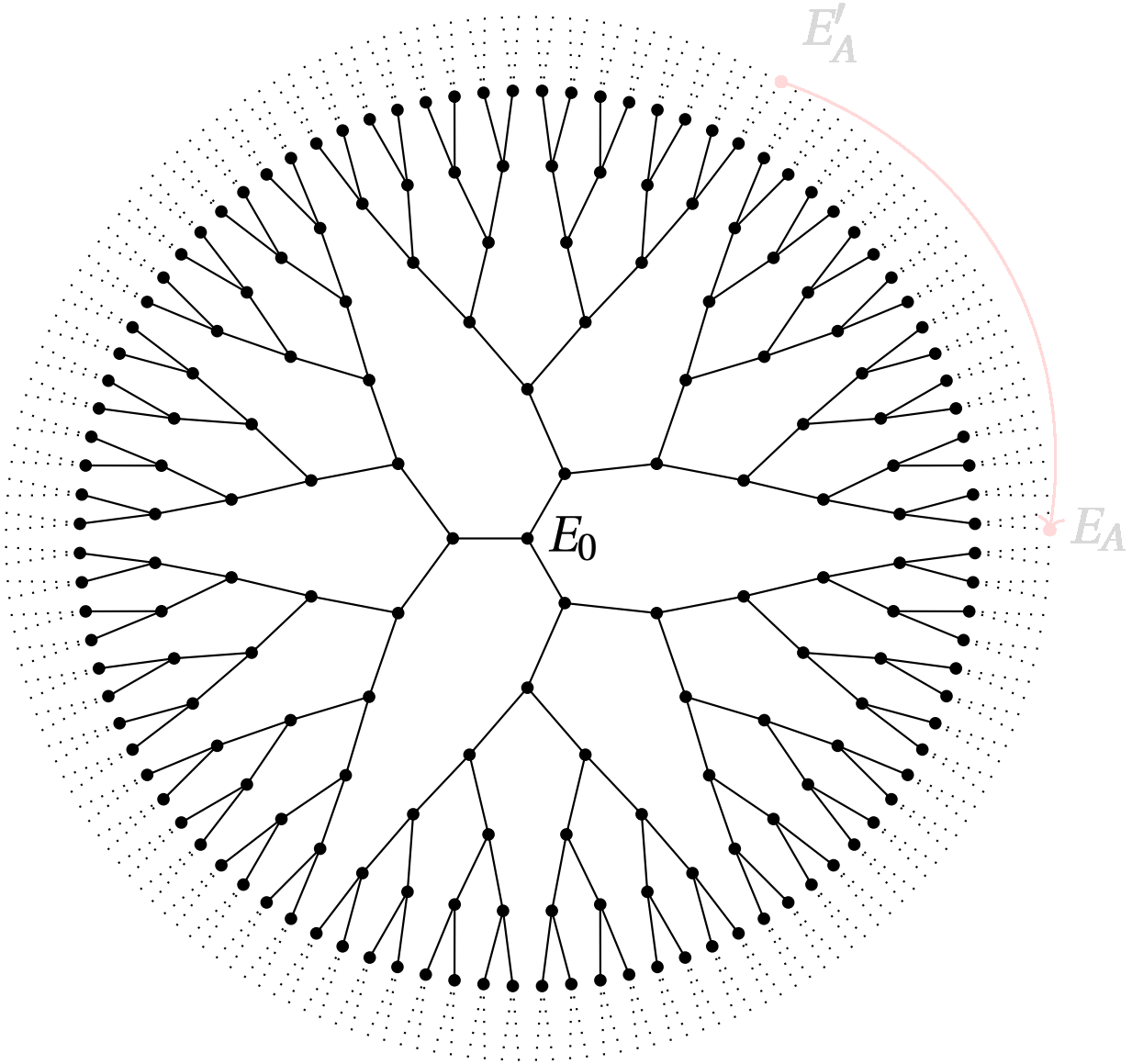
Let F_0, $F_1 : G \to X$ be two functions defined on some group G, such that there exists some $s \in G$ satisfying

$$F_0(g) = F_1(g \cdot s)$$

for all $g \in G$. The hidden shift problem is to find s given oracle access to the functions F_0 and F_1.

- If G is abelian and F_0, F_1 are injective, this can be solved in quantum subexponential time in $|G|$.

Intuition



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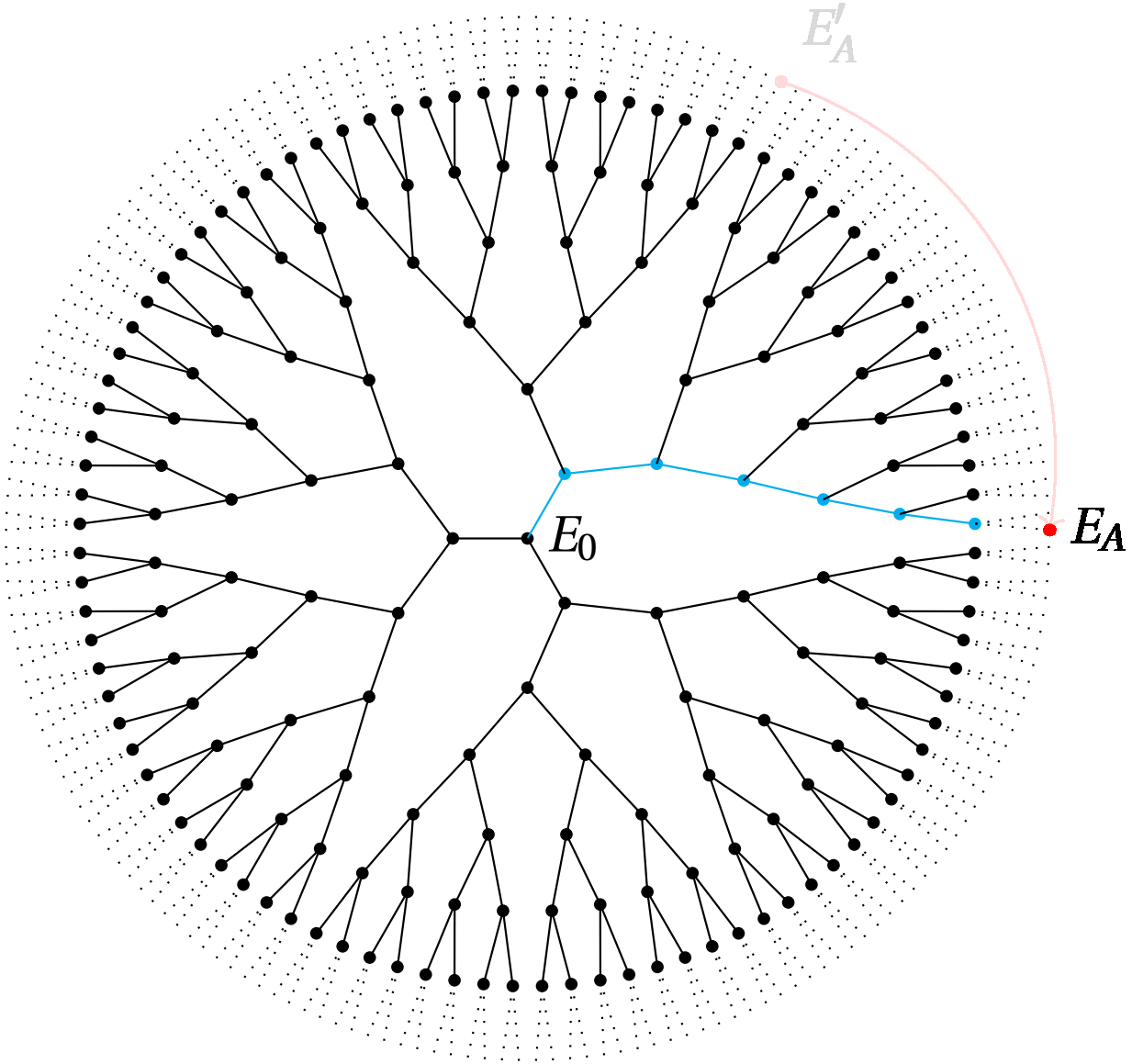

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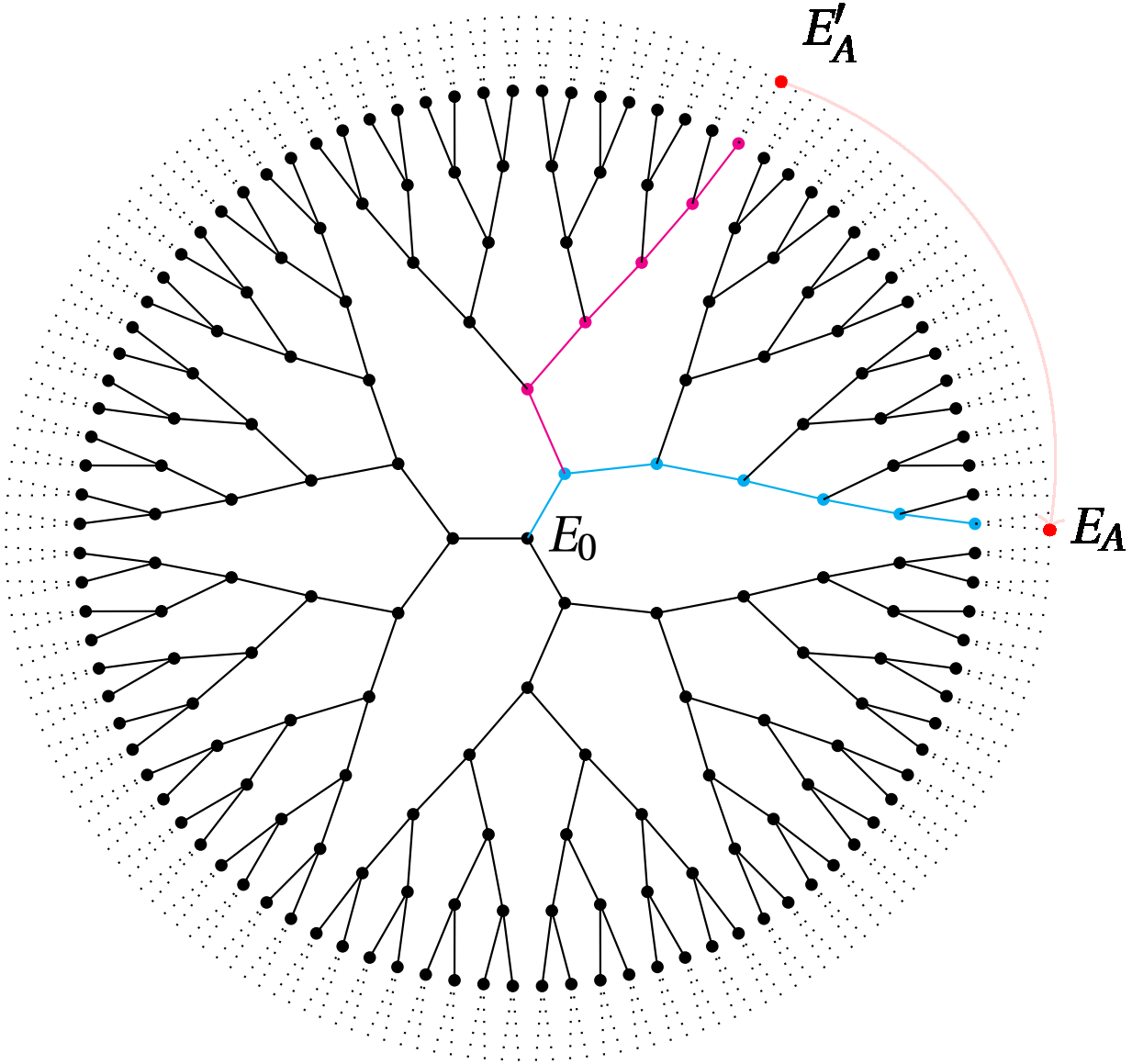

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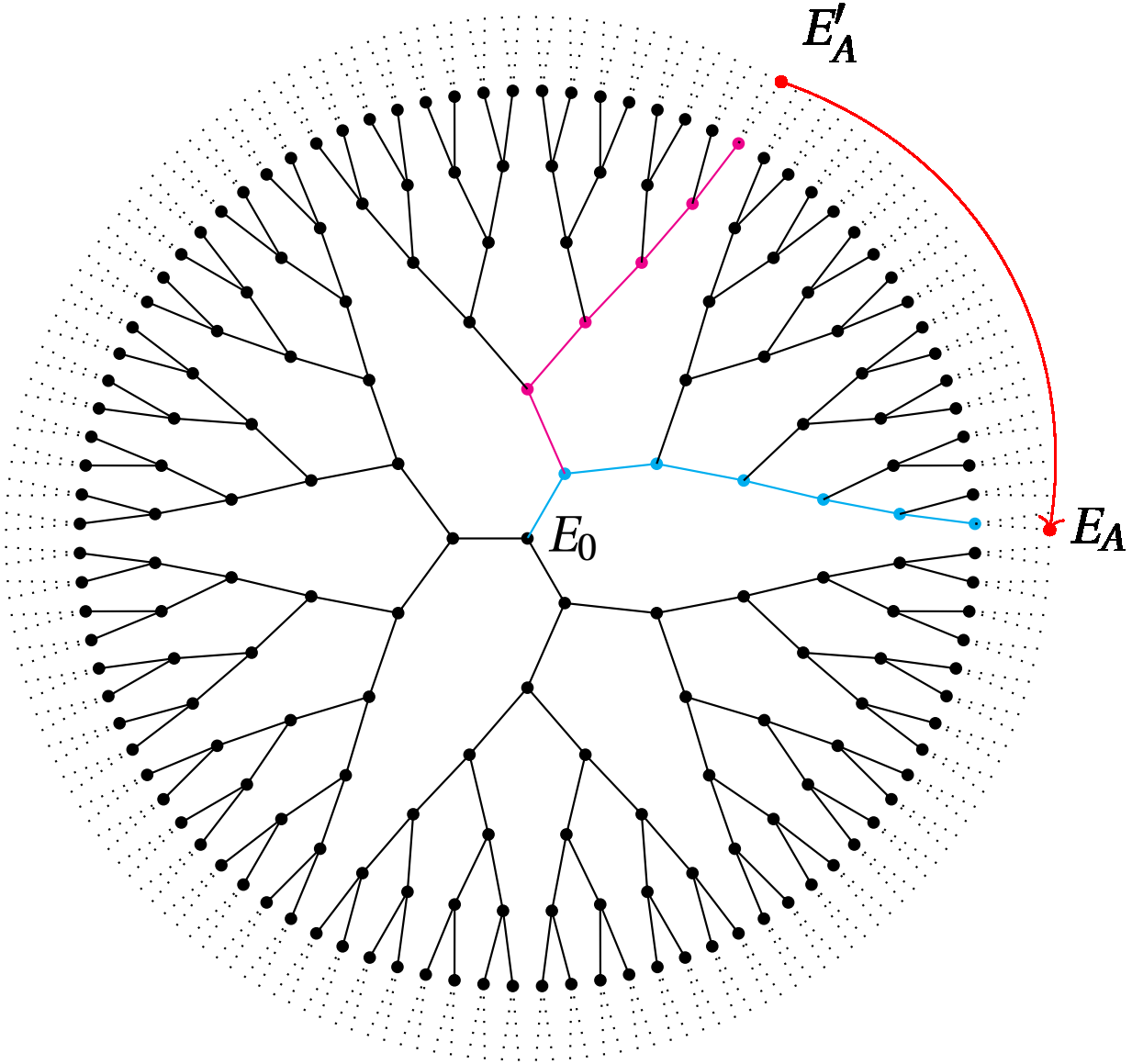

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Malleability Oracles

- Let $f : I \to O$ be an injective one-way function.
- Let G be a group acting on I.

Definition (Malleability oracles)

A *malleability oracle* for G at $o := f(i)$ provides $f(g \cdot i)$ for any $g \in G$, i.e., given any $g \in G$ and $f(i)$ the malleability oracle evaluates

$$g \mapsto f(g \cdot i).$$

Malleability oracles

Idea: Find preimage of $f(i)$ via a hidden shift computation.

Requirements:

1. Let G act transitively on I, and assume we have a malleability oracle for G at $f(i)$.
2. Let f be injective and let G be a finitely generated abelian group acting freely on I.

Strategy to compute i:

1. Pick any $j \in I$. (By transitivity of G, $\exists \sigma$ s.t. $i = \sigma \cdot j$.)
2. Define $F_0(g) := f(g \cdot j)$.
3. Define $F_1(g) := f(g \cdot i)$.
4. Compute σ such that $F_0(\sigma g) = F_1(g)$.
   (Injective abelian hidden shift problem which can be solved in quantum subexponential time.)
5. Compute $i = \sigma \cdot j$.

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OWF underlying SIDH

Let E_0 with $j(E_0) = 1728$ be the starting curve in SIDH with known endomorphism ring, and let N_1 and N_2 be the security parameters of Alice and Bob respectively.

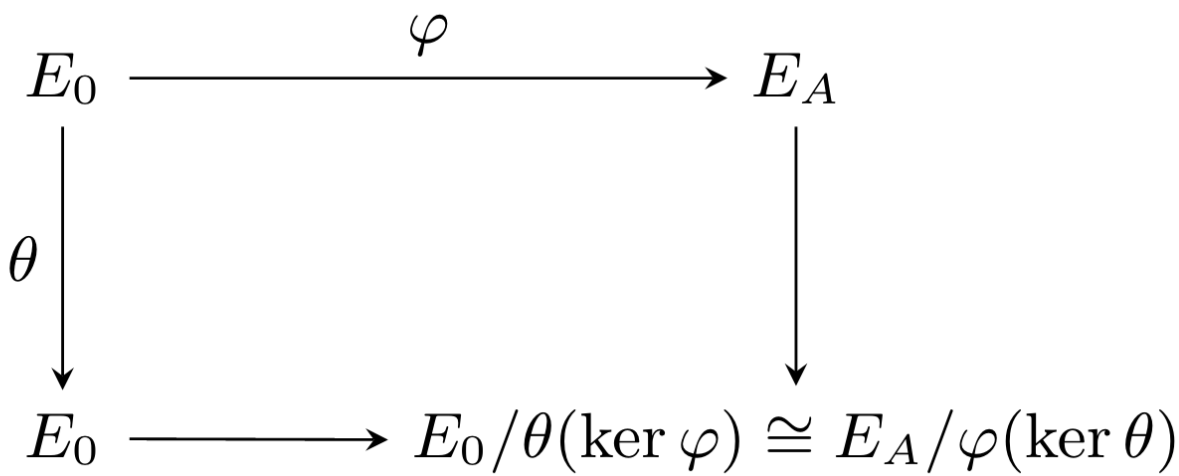
- $I :=$ {subgroups of order N_1 of E_0}
- $O :=$ {elliptic curves at distance N_1 from E_0}

$$f : I \to O$$
$$K \mapsto E_0/K \text{ and torsion point images}$$

Malleability oracle for SIDH

Let G be a multiplicative subgroup of $\left(\mathrm{End}(E_0)/N_1\mathrm{End}(E_0)\right)^*$, where representatives are endomorphisms of degree coprime to $\deg(\varphi)$.

Idea: Use torsion point information to construct malleability oracle at $E_A = f(K)$, i.e. given $\theta \in G$ compute $f(\theta \cdot K) = E_0/(\theta(K))$.



Lifting:

- For $\theta \in G$, find $\theta' \in \mathrm{End}(E_0)$ inducing the same action on I such that $\deg(\theta')|N_2$.
- Compute $\varphi(\ker\theta')$ using torsion point information.

Intermediate tasks

1. Partition I, i.e. cyclic subgroups of order N_1 of E_0, into (large) partitions such that f restricted to each one is injective.
2. Find abelian subgroups of $\left(\mathrm{End}(E_0)/N_1\mathrm{End}(E_0)\right)^*$ acting freely and transitively on each partition.
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Lifting of endomorphisms from $\pi\mathbb{Z}[\iota]$

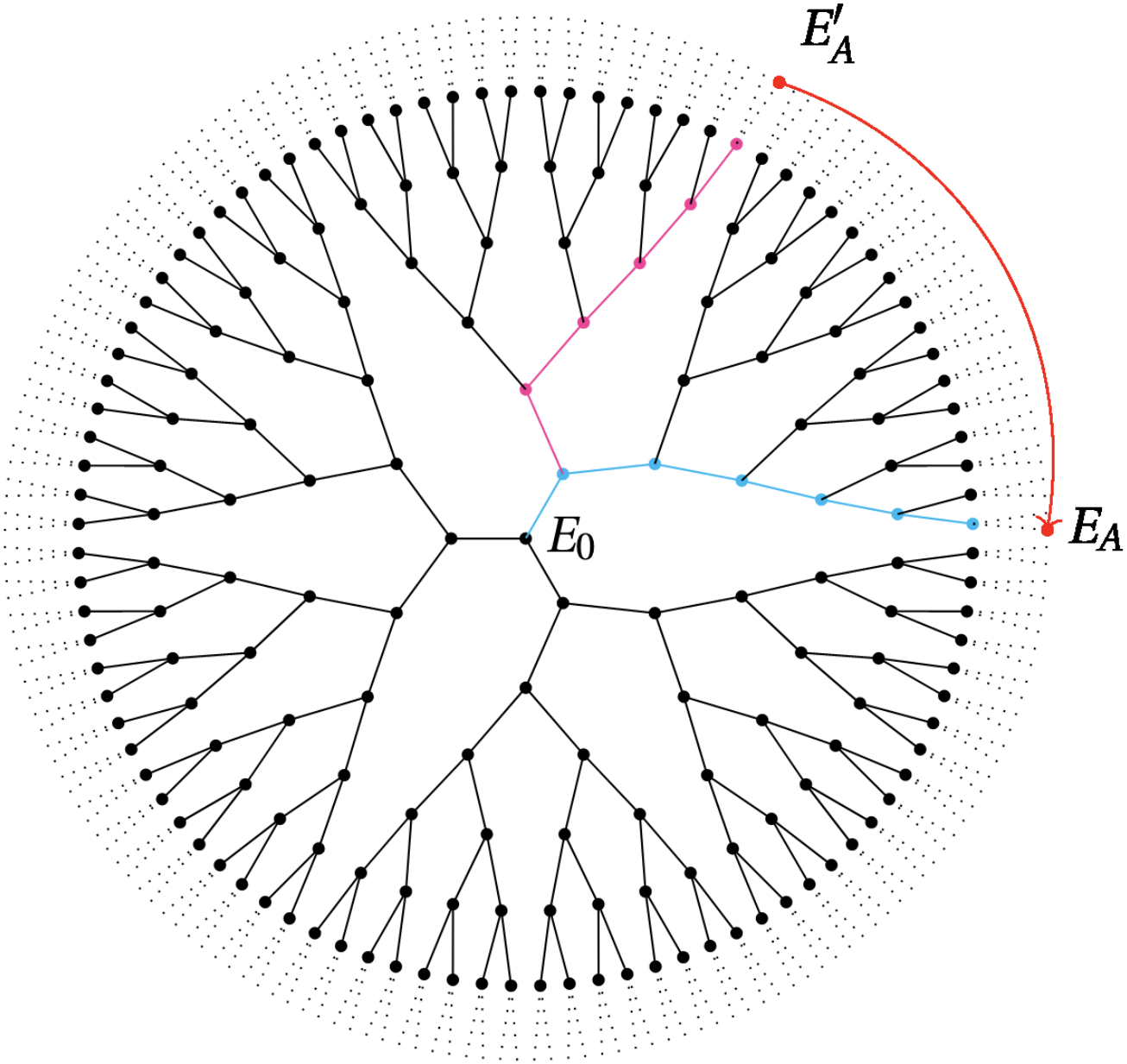
- Lifting algorithm: Given $\theta \in \pi\mathbb{Z}[\iota]$, return θ' of norm N_2 or eN_2 for some small e s.t. θ and θ' induce the same action on I.
- Works by solving a norm equation similar to the one in the KLPT algorithm.
- Our algorithm works for $N_2 > pN_1^4$.

Remark

For an improved lifting algorithm, we only expect solutions if

$$N_2 > pN_1^2.$$

Intuition continued



Summary

- For $N_2 > pN_1^4$, the problem underlying SIDH can be solved in quantum subexponential time via a reduction to the hidden shift problem.
- Despite SIDH's "non-commutative nature", there is an abelian group action on its key space.
- We embed the attack in a more general framework that also captures further attacks.
- SIKE or balanced SIDH parameters are not threatened by this attack, which breaks parameters that were already known to be insecure [DKL+21].

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