Cryptanalysis of an Oblivious PRF from Supersingular Isogenies

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March 2022

ISG Seminar

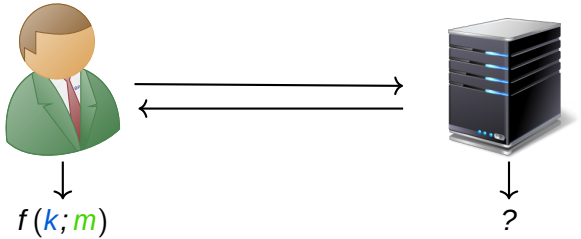
Content

- Definition of (V)OPRFs
- Applications
  - OPAQUE
  - PrivacyPass
- Isogenies and SIDH
- OPRF from isogenies
- Cryptanalytic results
  - Polytime and subexponential attacks
  - Requirement for trusted setup

Oblivious Pseudorandom Function (OPRF)

An OPRF is a two-party protocol to evaluate a PRF $f(k; m)$ where:

- The client learns $f(k; m)$, one evaluation of a PRF on a chosen input
- The server learns nothing about m



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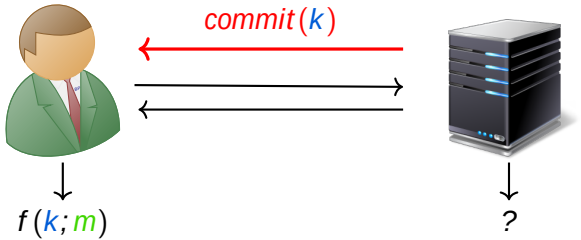


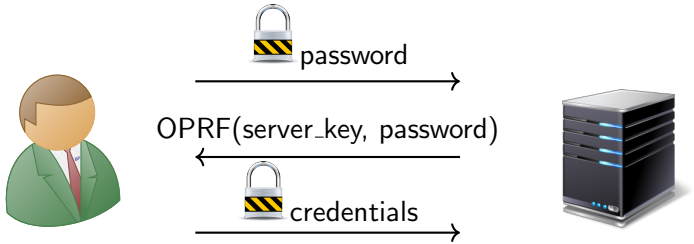
- An OPRF is called *verifable*, if the server proves to the client that output was computed using the key k

OPAQUE: OPRF + PAKE

- Use passwords that never leave your device

How to check a password that you have never seen?

Registration Phase:



OPAQUE: OPRF + PAKE

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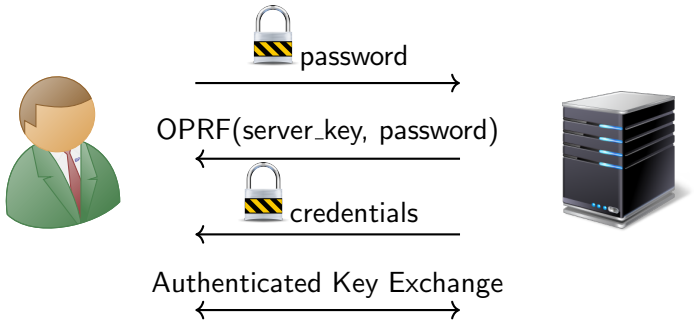
Login Phase:



PrivacyPass

- Generate cryptographically 'blinded' tokens that can be signed by server after client authenticates themselves (e.g. CAPTCHA solution)
- Security properties:
  1. Unlinkability
  2. Unforgeability
- Construction:
  - VOPRF for issuance of tokens during blind signing phase
  - Verification of anonymous tokens during redemption phase

Existing Constructions

Parameters: group G of order q, hash functions H_1, H_2 onto G and $f0;1g^{`}$ resp.

Client $C(m)$		Server $S(k)$
Pick r $_R$ Z_q		
Set a $(H_1(m))^r$	$\xrightarrow{a}$	
		If a 2 G; set b a^k
	$\xleftarrow{b}$	
If b 2 G; set v $b^{1=r}$		
Output $H_2(m;v)$		

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Post-quantum OPRF:

- Construction from lattices [ADDS19]
- Construction from isogenies [BKW20]

Isogenies

Let E_0, E_1 be elliptic curves defined over a field $\overline{\mathrm{F}}_p$

- An *Isogeny* is non-constant rational map $' : E_0 \; ! \; E_1$ that is also a group homomorphism
- The kernel of an isogeny determines the image curve up to isomorphism ($E_0 = \ker(') := E_1$)
- Two curves E_0, E_1 are isomorphic if and only if they have the same j-invariant
- (Separable) isogenies correspond to subgroups of E_0 (order of subgroup equals degree of isogeny)

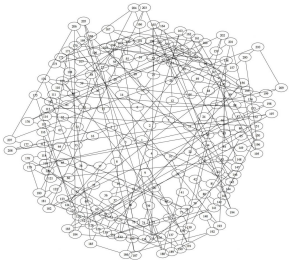

Figure: Image by D. Charles

SIDH [JD11]

- Fix a prime p such that $p = N_1 N_2 \quad 1$, E_0=$\mathbb{F}_p^2$ and bases $\langle P_A, Q_A \rangle = E_0[N_1]$, $\langle P_B, Q_B \rangle = E_0[N_2]$

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- Alice's secret is $A := P_A + [\mathrm{sk}_A]Q_A$
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- Alice sends E_A, $\quad_A(P_B)$, $\quad_A(Q_B)$
- Bob sends E_B, $\quad_B(P_A)$, $\quad_B(Q_A)$

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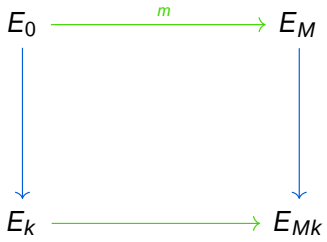
- Alice sends $E_A, \quad _A(P_B), \quad _A(Q_B)$
- Bob sends $E_B, \quad _B(P_A), \quad _B(Q_A)$

- The shared secret is the j-invariant of E_{AB}

Oblivious Pseudorandom Functions from Isogenies [BKW20]

Client
Server



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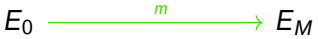

Client
Server

E_k E_{Mk}

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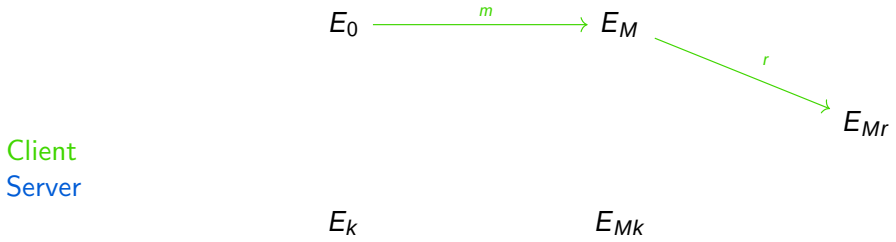

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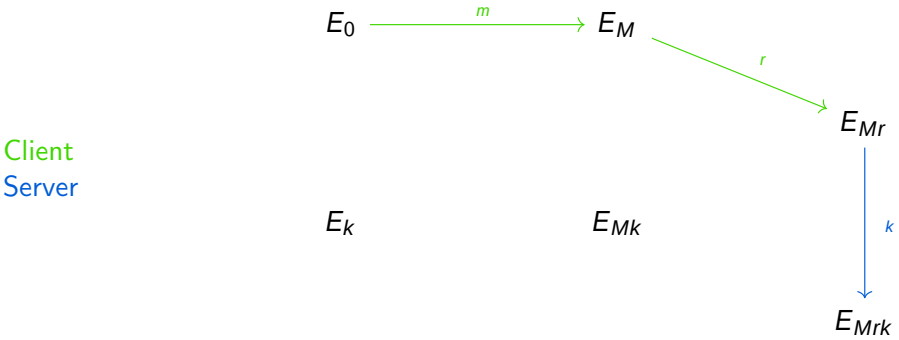

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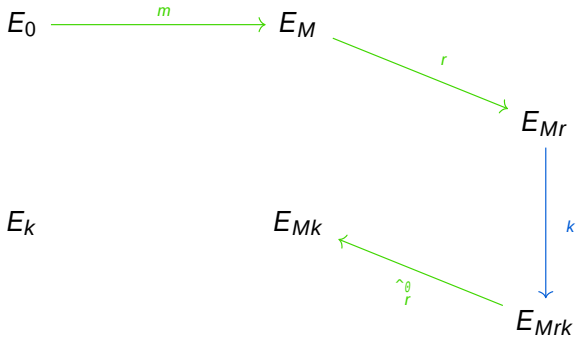

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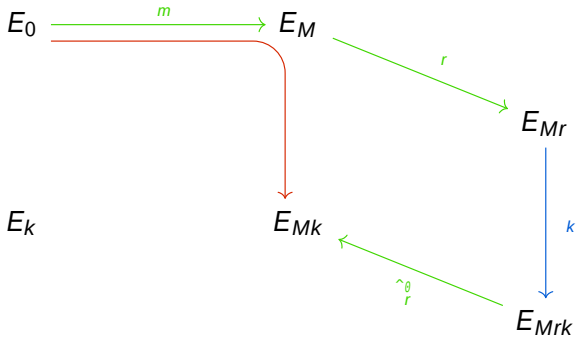
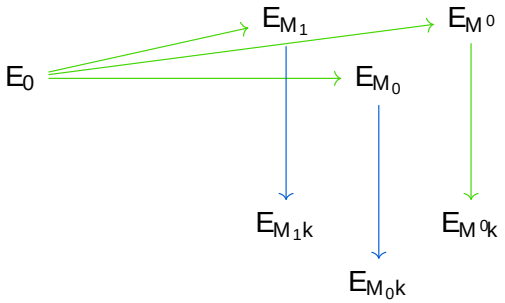


$$f(k;m) = H(m; j(E_{Mk}); \mathrm{pk})$$

Pseudorandomness of an Oblivious PRF

- An attacker should not be able to evaluate the OPRF without the server's help even after multiple queries



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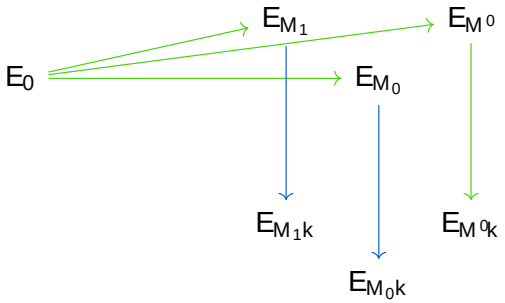

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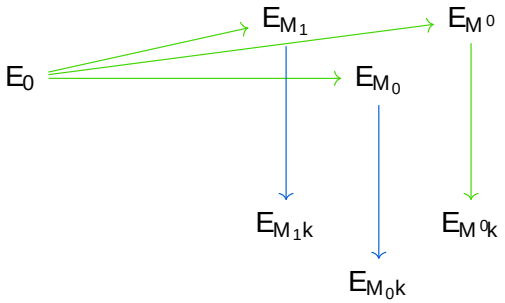

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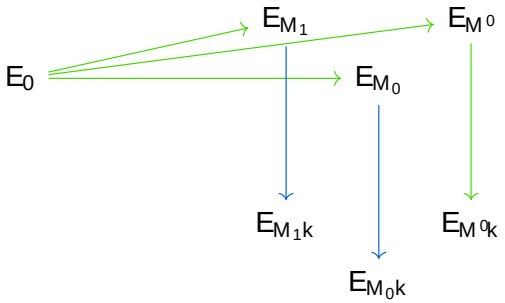
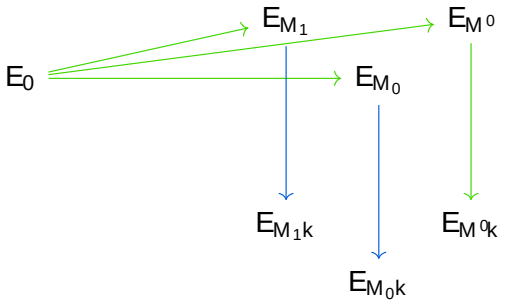
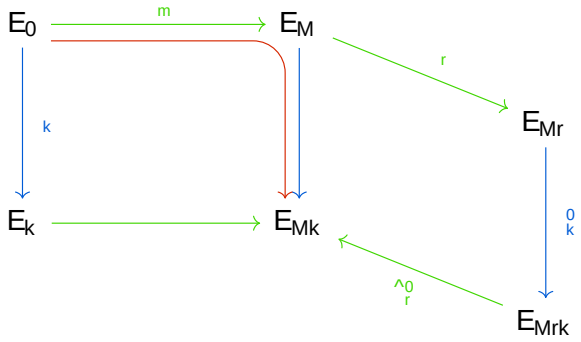


Pseudorandomness of an Oblivious PRF

- An attacker should not be able to evaluate the OPRF without the server's help even after multiple queries
- Pseudorandomness of [BKW20] is based on a new `auxiliary one-more' assumption



Attacking the `one-more' Assumption



- Find E_k and $h_k(M)i$ for some point $M \in E_0[2^n]$

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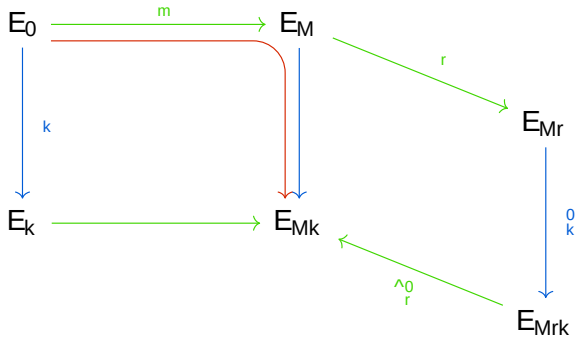

- Find E_k and $\langle \phi_k(M) \rangle$ for some point $M \in E_0[2^n]$
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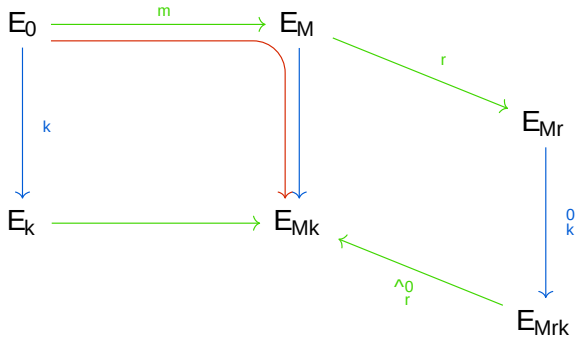
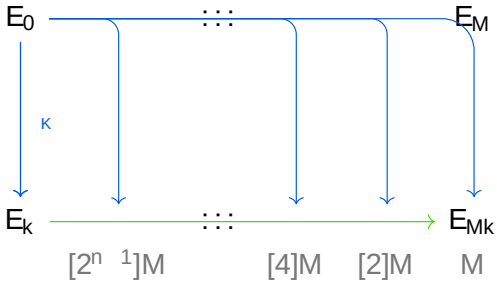


- Find E_k and $\langle_k(M)\rangle$ for some point $M \in E_0[2^n]$
- Combine multiple points to obtain $\langle_k(E_0[2^n])$ up to scalar multiplication
- Given point $P \in E_0[2^n]$, compute $\langle_k(P)\rangle$ and finally $E_k = \langle_k(P)\rangle = E_{Pk}$

A Polytime Attack

Recovering points on E_k



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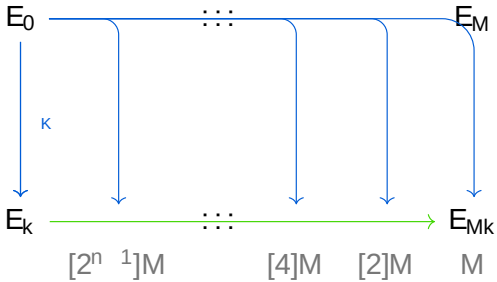

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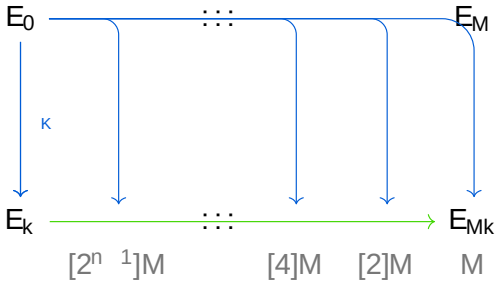

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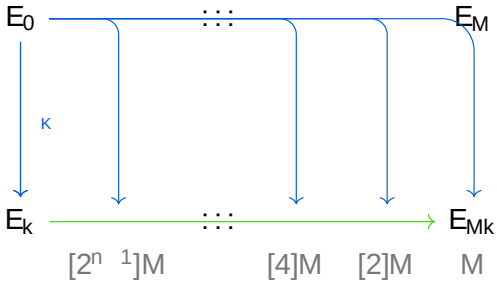

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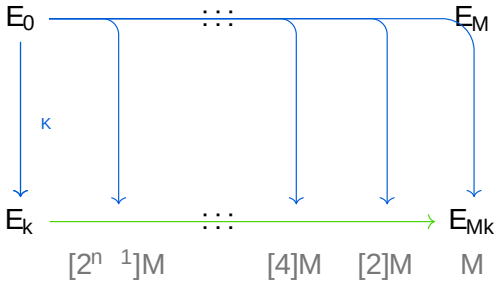

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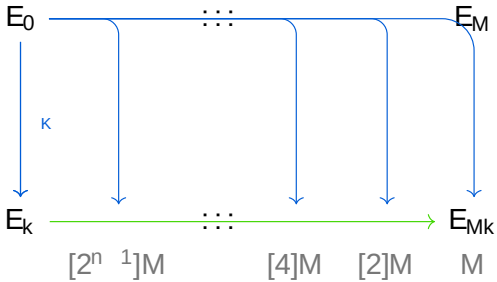

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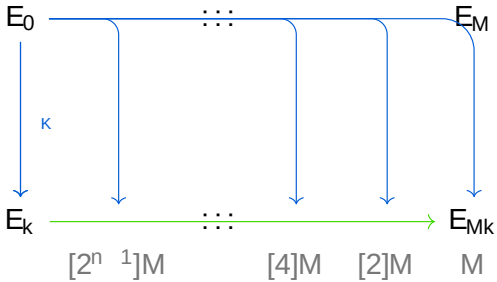

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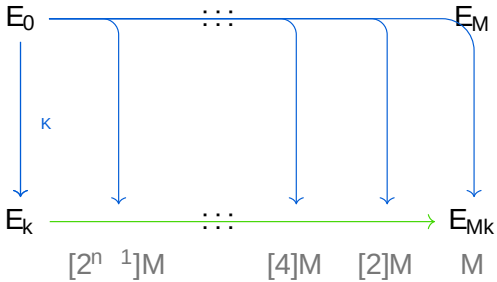

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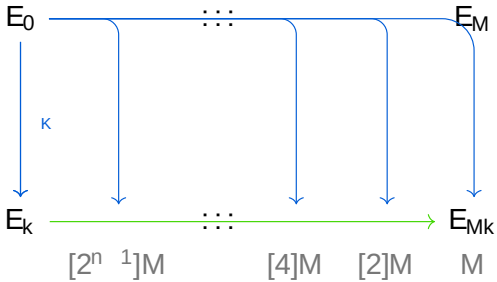

$$\ker = h_K(M)i$$

A Polytime Attack

Combining the points

Given M on $E_0[2^n]$, we can recover $\langle K(M) \rangle$

A Polytime Attack

Combining the points

Given M on $E_0[2^n]$, we can recover $\langle \phi_K(M) i \rangle$) we can recover $[] \phi_K(M)$

A Polytime Attack

Combining the points

Given M on $E_0[2^n]$, we can recover $\langle [\ ]_K(M) \rangle$) we can recover $[\ ]_K(M)$

We query on $M; N; M + N$ and obtain

$$M^0 = [\ ]_K(M)$$
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$$R^0 = [\ ]_K(M + N) = [a]M^0 + [b]N^0$$

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A Polytime Attack

Combining the points

Given M on $E_0[2^n]$, we can recover $\langle \phi_K(M) \rangle$ $\Rightarrow$ we can recover $[\,]\,\phi_K(M)$

We query on $M; N; M + N$ and obtain

$$\left.\begin{aligned} M' &= [\,]\,\phi_K(M) \\ N' &= [\,]\,\phi_K(N) \\ R' &= [\,]\,\phi_K(M+N) = [a]M' + [b]N' \end{aligned}\right\} \Rightarrow -\frac{b}{a}$$

Breaking the assumption

Given any $P = [x]M + [y]N$, we can compute $\langle \phi_K(P) \rangle = \langle [x]M' + [y]N' \rangle$

A Polytime Attack

Results

- $O(\ )$ queries recover $\langle_K(M)\rangle$ for any M in $E_0[2^n]$
- With three subgroups, we can compute $\langle_K(P)\rangle$ for any P without further interactions
- This breaks the `one-more' assumption

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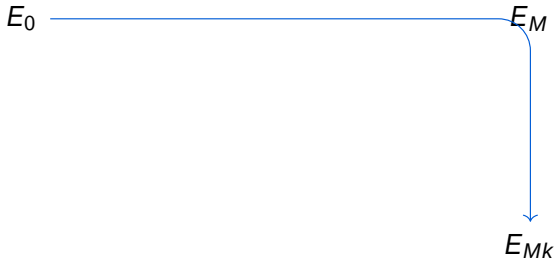
But

- It is easy to check that query points have full order

A Subexponential Attack

Using full-order queries



A Subexponential Attack

Using full-order queries

E_0

$\longleftarrow E_M$

E_{Mk}

A Subexponential Attack

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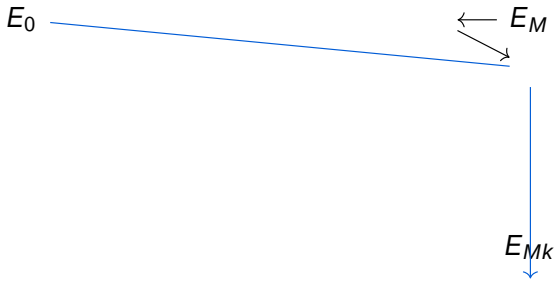
Using full-order queries



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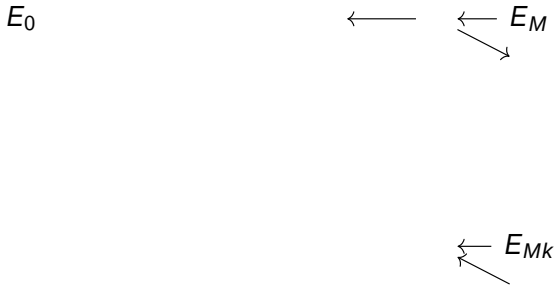

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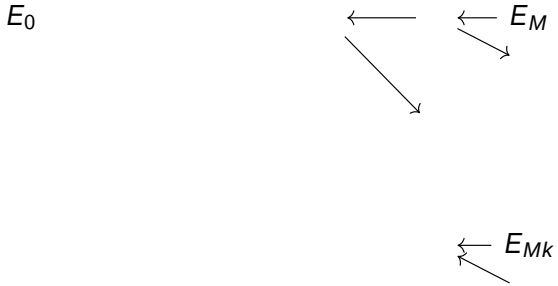

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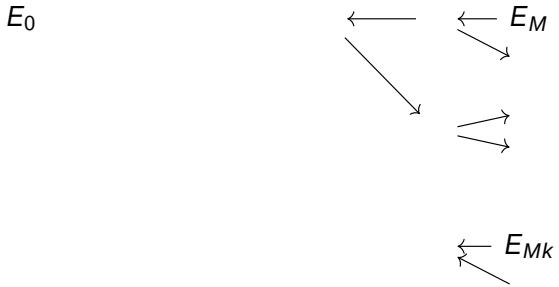

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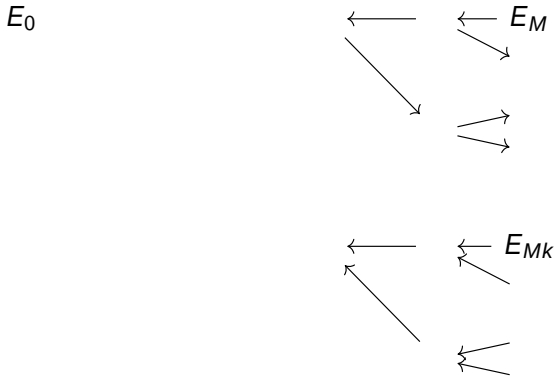

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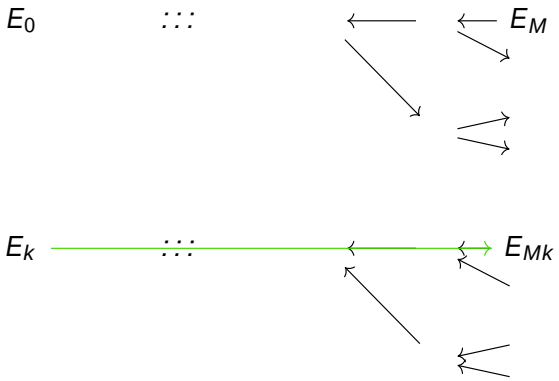
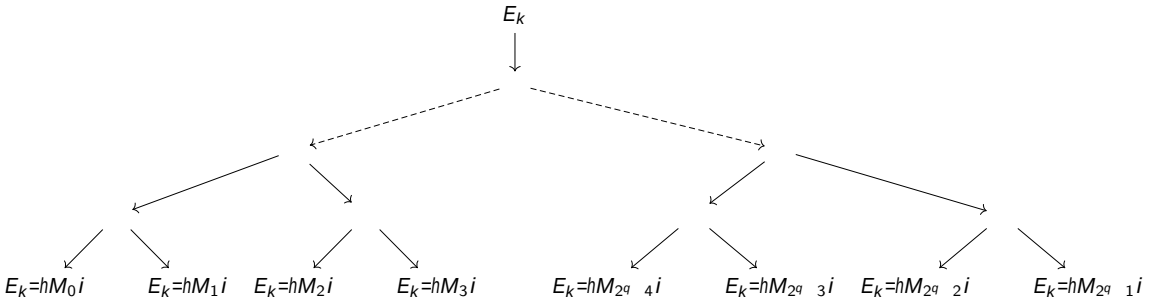


A Subexponential Attack

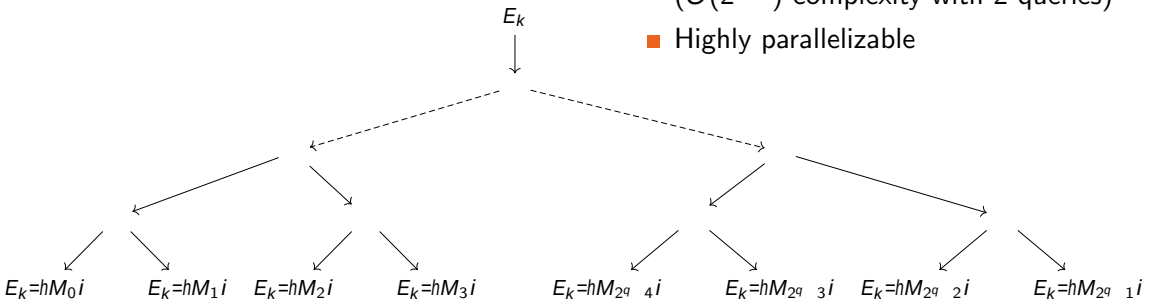
Building a tree



A Subexponential Attack

Building a tree

- Queries/complexity trade-offs ($O(2^{=3})$ complexity with 2 queries)
- Highly parallelizable



A Subexponential Attack

The full attack:

- Use the binary tree to recover points on E_k
- Second part of the attack same as polytime attack
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Countermeasures:

- No obvious countermeasures
- Increase the parameter size?) very large degrees
- New efficient solutions?

Implementation Results

Parameters				MITM		Running Time
$\log p$		n	q	Distance	Memory (kB)	(s)
112	8	20	3	8	3.5	15
216	16	40	6	10	13.8	212 (3.53 m)
413	32	80	8	16	211.4	1,371 (22.85 m)
859	67	169	11	26	14,073	163,869 (1.89 d)
1,614	128	320	18	40	3,384,803	*174,709,440 (5.54 y)*

Available at `https://github.com/isogenists/isogeny-OPRF`

The Starting Curve

Who chooses E_0?

- The client
- A third-party
- The server
- Known curve ($j(E_0) = 1728$)
- Trusted setup

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- **Trusted setup**

Conclusion

- Two attacks on 'one-more' assumption and the pseudorandomness of Boneh et al.'s OPRF
- A proof of concept implementation of the attack
- Need for a trusted setup
- CSIDH-based OPRF construction is not affected by the attack

Paper available at `https://ia.cr/2021/706`

References

[BKW20] Dan Boneh, Dmitry Kogan, and Katharine Woo. Oblivious pseudorandom functions from isogenies. In *Advances in Cryptology - ASIACRYPT 2020 - 26th International Conference on the Theory and Application of Cryptology and Information Security, Daejeon, South Korea, December 7-11, 2020, Proceedings, Part II*, pages 520–550, 2020.