SCALLOP: a somewhat scalable effective group action from isogenies

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February 2024

Isogeny Club

Cryptographic group actions

Definition

A group action of a group G on a set X is a function

$$\star : G \times X \to X$$

- $e \star x = x$
- $(gh) \star x = g \star (h \star x)$

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- Parallelization prob.: given $x, g \star x, h \star x$, find $(gh) \star x$

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- Typically group action-based cryptography has focussed on group actions that are both free and transitive

EGA: effective group action

Definition (EGA)

A group action $(G, X, \star)$ is effective, if there exist efficient (PPT) algorithms for

- membership testing, equality testing, sampling and computing the operation and inversion in G
- membership testing and unique representation in X
- computing $g \star x$ for any $g \in G$ and $x \in X$.

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CSIDH is not an EGA!

For arbitrary $g \in G$ and $x \in X$, computing $g \star x$ is not efficient!

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- Fix list of elements $\mathfrak{l}_1, \ldots, \mathfrak{l}_n$ spanning G such that $\mathfrak{l}_i \star E$ can be efficiently evaluated for every $E \in X$
- Can evaluate $\prod_i \mathfrak{l}_i^{e_i} \star E$ for $E \in X$ efficiently as long as exponents $(e_1, \ldots, e_n) \in \mathbb{Z}^n$ are sufficiently small, i.e. e_i sampled from $[-B, B]$ for some bound B in CSIDH

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So what?

EGA vs REGA: Identification protocols and Fiat-Shamir signatures

Let $(G, X, \star)$ be an EGA. Zero-knowledge proof of knowledge of secret $s \in G$ corresponding to public key $(E_0, E_1 := s \star E_0) \in X^2$:

- Prover commits to $E_c := r \star E_0$ for random $r \in G$

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Can turn protocol into (non-interactive) signature scheme with Fiat-Shamir transform.

- Zero-knowledge proof breaks for REGA, $s^b r^{-1}$ can leak information about s
- Fix: rejection sampling (see SeaSign) $\Rightarrow$ considerable increase in parameters, much less efficient

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Online phase to evaluate $\mathfrak{l}_1^e \star E$ (for all $e \in \mathbb{Z}$):

- Solve (approximate) CVP of $(e, 0, \ldots, 0)$ in $\mathcal{L}$ to find decomposition $\mathfrak{l}_1^e = \prod_i \mathfrak{l}_i^{e_i}$ with small exponents e_i
- Evaluate the restricted group action $\prod_i \mathfrak{l}_i^{e_i} \star E$

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Caution

Depending on the group G, the precomputation might be computationally infeasible!

CSI-FiSh signature scheme [BKV19]

- Based on group action of CSIDH-512
- Precompute lattice of relations $\mathcal{L}$ for the generators used in CSIDH-512 using an index-calculus approach
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Caution

Computing the structure of the acting group for larger CSIDH parameters is infeasible with currently known algorithms.

Idea

Motivation

Introduce group action that solves the scaling issue of CSI-FiSh

(to some extent..)

Cryptographic group actions $(G, X, \star)$ for which structure of G can be computed more easily?

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Idea

Can compute class number $|\mathrm{Cl}(\mathfrak{O})|$ for $\mathfrak{O}$ of the form $\mathbb{Z} + f\mathfrak{O}_0$ from class number $|\mathrm{Cl}(\mathfrak{O}_0)|$ and factorization of f.

Let $f \in \mathbb{Z}$, let $\mathfrak{O}_0$ be a quadratic order of class number h_0 and discriminant d_0 and let $u_0 := |\mathfrak{O}^\times|/2$.
For $\mathfrak{O}$ of the form $\mathbb{Z} + f\mathfrak{O}_0$ we have

$$|\mathrm{Cl}(\mathfrak{O})| = \left(f - \left(\frac{d_0}{f}\right)\right)\frac{h_0}{u_0}.$$

Oriented elliptic curves

Let $\mathfrak{O}$ be an imaginary quadratic order, e.g. $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-p}]$, in an imaginary quadratic field K.

Definition

For any elliptic curve E, a K-orientation is a ring homomorphism $\iota : K \to \text{End}(E) \otimes \mathbb{Q}$. A K-orientation induces a primitive $\mathfrak{O}$-orientation if $\iota(\mathfrak{O}) = \text{End}(E) \cap \iota(K)$. In that case, the pair (E, ι) is called an $\mathfrak{O}$-oriented curve.

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- ι embeds $\mathfrak{O}$ into $\mathrm{End}(E)$ (and no superorder of $\mathfrak{O}$)
- We will represent the orientation by a kernel representation of an endomorphism corresponding to a generator of $\mathfrak{O}$

Group actions on oriented curves

- Let X be the set of primtively $\mathfrak{O}$-oriented curves (E, ι) up to isomorphism and Galois conjugacy

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$$\mathrm{Cl}(\mathfrak{O}) \times X \to X$$

- Group action is free and transitive (see [Onu21])
- Example: CSIDH, where $\mathfrak{O} = \mathbb{Z}[\sqrt{-p}]$ with orientations that send $\sqrt{-p}$ to Frobenius endomorphisms

Group actions on oriented curves cont.

- Computing group action using isogenies:
  - Let $\mathfrak{a} \subset \mathfrak{O}$ ideal, (E, ι_E) an elliptic curve with $\mathfrak{O}$-orientation
  - Define $E[\mathfrak{a}] = \bigcap_{\alpha \in \mathfrak{a}} \ker \iota_E(\alpha)$ and let

$$\varphi_{\mathfrak{a}}^E := E \to E_{\mathfrak{a}} := E/E[\mathfrak{a}] \quad \text{and} \quad \iota_{E_{\mathfrak{a}}}(x) = \frac{1}{n(\mathfrak{a})}\varphi_{\mathfrak{a}}^E \circ \iota(x) \circ \hat{\varphi}_{\mathfrak{a}}^E$$

  - $\mathfrak{a} \star (E, \iota_E) = (E_{\mathfrak{a}}, \iota_{E_{\mathfrak{a}}})$

Computing with oriented curves

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CSIDH:

- Ideal $\mathfrak{l}_i \subset \mathbb{Z}[\sqrt{-p}]$ acts through an isogeny of degree $\ell_i = n(\mathfrak{l}_i)$ whose kernel is stabilized by the Frobenius endomorphism π corresponding to $\sqrt{-p}$
- To compute $\mathfrak{l}_i \star E$ it is sufficient to evaluate the Frobenius endomorphism π on $E[\ell_i]$ and determine its eigenspaces

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~~CSIDH~~ General:

- Ideal $\mathfrak{l}_i \subset$ ~~$\mathbb{Z}[\sqrt{-p}]$~~ $\mathfrak{O}$ acts through an isogeny of degree $\ell_i = n(\mathfrak{l}_i)$ whose kernel is stabilized by ~~the Frobenius endomorphism π corresponding to $\sqrt{-p}$~~ endomorphism ω corresponding to a generator of $\mathfrak{O}$
- To compute $\mathfrak{l}_i \star E$ it is sufficient to evaluate ~~the Frobenius endomorphism π~~ endomorphism ω on $E[\ell_i]$ and determine its eigenspaces
- Compute (kernel) representation of endomorphism corresponding to generator of $\mathfrak{O}$ under orientation

Wishlist

To compute the class group structure, we want:

- $|\mathrm{Cl}(\mathfrak{O}_0)|$
- $\mathfrak{O} = \mathbb{Z} + f\mathfrak{O}_0$ such that factorisation of conductor f known
- $|\mathrm{Cl}(\mathfrak{O})|$ smooth enough to be able to compute the lattice of relations between ideal actions

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To represent and compute with oriented curves explicitly, we want:

- A generator α of $\mathfrak{O}$ of smooth norm $L_1^2 L_2$ to efficiently compute and represent corresponding endomorphisms
- A primitively $\mathfrak{O}$-oriented starting curve

SCALLOP: Precomputation

SCALable isogeny action based on Oriented supersingular curves with Prime conductor

- Take $\mathfrak{O}_0$ with $|\mathrm{Cl}(\mathfrak{O}_0)| = 1$, we take $\mathfrak{O}_0 = \mathbb{Z}[i]$

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SCALable isogeny action based on Oriented supersingular curves with Prime conductor

- Take $\mathfrak{O}_0$ with $|\mathsf{Cl}(\mathfrak{O}_0)| = 1$, we take $\mathfrak{O}_0 = \mathbb{Z}[i]$
- Generate candidates for $\mathfrak{O}$ with smooth generator until
  - conductor $f \approx 2^{2\lambda}$ is prime (avoids factoring f)
  - class number $|\mathsf{Cl}(\mathfrak{O})|$ is reasonably smooth

SCALLOP: Precomputation (contd.)

- Fix $\ell_1, \ldots, \ell_n$ to be the smallest n split primes in $\mathbb{Z}[i]$, e.g. $(5) = (2+i)(2-i)$, $(13) = (3+2i)(3-2i)$ etc.

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- Randomly pick signs for ideals (or their squares) above ℓ_i and consider product of generators $\Rightarrow$ smooth norm $L_1^2 L_2$ by construction, i.e. generator corresponds to endomorphism with kernel representation points of order L_1 and $L_1 L_2$

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- Test primality of conductor f of product, then compute corresponding class number and test smoothness using ECM factoring with abort
- Asymptotically, $L_f(1/2)$ search for $L_f(1/2)$-smooth $|\mathrm{Cl}(\mathfrak{O})|$

SCALLOP: Precomputation (contd.)

- Choose prime characteristic p to maximise efficiency of evaluating the group action (and large enough to prevent attacks), i.e. take $p = \prod_i \ell_i \pm 1$

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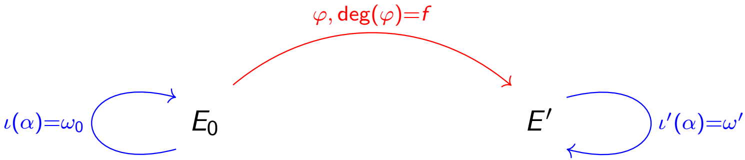
- Compute reduced basis of $\mathcal{L}$ using BKZ as in CSI-FiSh
- Generate a starting curve with $\mathfrak{O}$-orientation

Precomputation: Generation of starting curve



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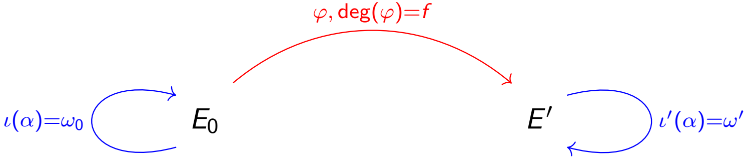
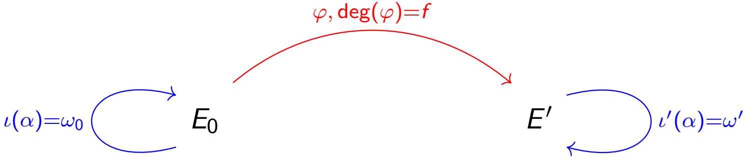


Given characteristic p and large prime f with $\mathfrak{O} = \mathbb{Z} + f\mathfrak{O}_0 = \mathbb{Z}[\alpha]$ for some α of norm $L_1^2 L_2$. How to compute effective primitive $\mathfrak{O}$ orientation (E', ι')?

- Push kernel of ω_0 through φ, but deg(f) large prime
  $\Rightarrow$ can't use Vélu's formulae

Precomputation: Generation of starting curve



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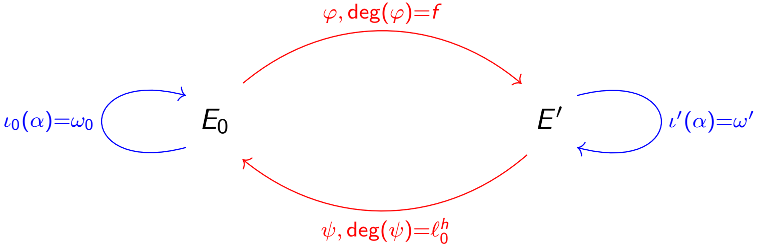

- $\mathfrak{O}_0$ special extremal order (see [KLPT14]) $\Rightarrow$ can find $\gamma \in \mathfrak{O}_0$ of norm M efficiently as soon as $M > p$
- Let ℓ_0 small prime not dividing $L_1 L_2$ and $h \in \mathbb{Z}$ such that $\ell_0^h > p/f$ and compute $\gamma \in \mathfrak{O}_0$ of norm $f\ell_0^h$ whose ideal corresponds to endomorphism $\psi \circ \varphi$

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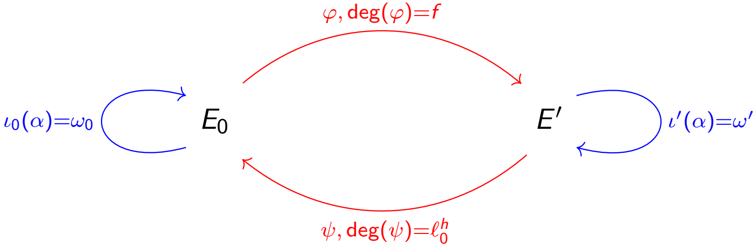

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- Push kernel of ω_0 through $\psi \circ \varphi$ (see e.g. [FKMT22]), brute-force ψ and compute ω'

SCALLOP: Online phase

- Generator of smooth norm of $\mathfrak{O}$ corresponds to endomorphism ω_E of smooth degree which we represented by kernels of two isogenies
- ω_E stabilizes kernels of isogenies used to compute group action

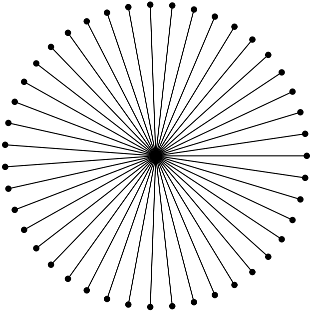

Figure: Isogeny volcano for $\mathfrak{O}$-oriented curves in SCALLOP.

SCALLOP: Online phase

- Generator of smooth norm of $\mathfrak{O}$ corresponds to endomorphism ω_E of smooth degree which we represented by kernels of two isogenies
- ω_E stabilizes kernels of isogenies used to compute group action
- Evaluate group action by transporting explicit orientation along the group action
- Computing explicit orientation leads to slowdown compared to CSI-FiSh with canonical orientation

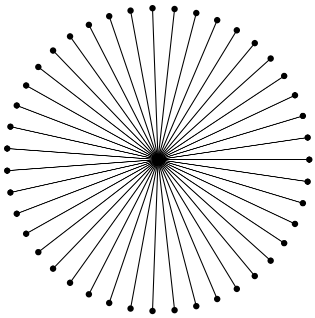

Figure: Isogeny volcano for $\mathfrak{O}$-oriented curves in SCALLOP.

Effective Group Actions: CSI-FiSh vs SCALLOP

<u>CSI-FiSh</u>

- $\mathfrak{O} = \mathbb{Z}[\sqrt{-p}]$

<u>SCALLOP</u>

- $\mathfrak{O} = \mathbb{Z} + f\mathfrak{O}_0$, f prime

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CSI-FiSh

- $\mathfrak{O} = \mathbb{Z}[\sqrt{-p}]$
- Expensive class group computation, only feasible for CSIDH-512 parameters

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- $|\mathrm{Cl}(\mathfrak{O})|$ free, sieve until smooth enough to compute lattice of relations

Effective Group Actions: CSI-FiSh vs SCALLOP

CSI-FiSh

- $\mathfrak{O} = \mathbb{Z}[\sqrt{-p}]$
- Expensive class group computation, only feasible for CSIDH-512 parameters
- Evaluation of group action with implicit orientation
- Online phase fast

SCALLOP

- $\mathfrak{O} = \mathbb{Z} + f\mathfrak{O}_0$, f prime
- $|\mathrm{Cl}(\mathfrak{O})|$ free, sieve until smooth enough to compute lattice of relations
- Need to compute explicit orientation along group action
- Online phase slower, but feasible for larger security levels

Implementation

Proof of concept implementation in C++ available at:
https://github.com/isogeny-scallop/scallop

- Concrete instantiation for SCALLOP matching the security levels of CSIDH-512 and CSIDH-1024
- Public keys of size roughly 1600bits for SCALLOP-512 and 2300bits for SCALLOP-1024

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- Concrete instantiation for SCALLOP matching the security levels of CSIDH-512 and CSIDH-1024
- Public keys of size roughly 1600bits for SCALLOP-512 and 2300bits for SCALLOP-1024
- Evaluation of the group action takes about 35 seconds for the smaller and 12.5 minutes for the larger parameter set
- Implementation shows feasibility, but further work needed to make the group action practical

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- Concrete instantiations of class group action using action of class group of imaginary quadratic order with large prime conductor f inside an imaginary quadratic field of small discriminant (SCALLOP)
- This instantiates effective group actions for security levels previously out of reach

Summary

- Provide framework to evaluate a new family of group actions on oriented elliptic curves via isogenies
- Concrete instantiations of class group action using action of class group of imaginary quadratic order with large prime conductor f inside an imaginary quadratic field of small discriminant (SCALLOP)
- This instantiates effective group actions for security levels previously out of reach
- Can build schemes that require to uniquely represent and efficiently act by arbitrary group elements for larger security levels than with CSIDH-512 group action

Questions

Open

- How to make group action evaluation faster?
- How to resolve the scaling issues of SCALLOP?

Questions

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Thank you!

More details:
ia.cr/2023/058

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